

## Minimal hypersurfaces in $\mathbb{R}^n$ as regular values of a function\*

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**Abstract.** In this paper we prove that if  $M = f^{-1}(0)$  is a minimal hypersurface of  $\mathbb{R}^n$ , where  $f : V \subset \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth function defined on a open set  $V$ , then  $f$  must satisfy the equation  $|\nabla f|^2 \Delta f = \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle$  for every  $x \in M$ . We will also prove that if  $M$  is the zero level set of a homogeneous 2 polynomial, then  $M$  must be a Clifford minimal hypersurface.

### 1. Introduction and preliminaries

In this paper we will consider hypersurfaces  $M \subset \mathbb{R}^n$  that are level sets of functions, i.e. we will assume that  $M = \{x \in V : f(x) = 0\}$  where  $f : V \rightarrow \mathbb{R}$  is a smooth function defined in an open set  $V$  of  $\mathbb{R}^n$  and  $|\nabla f(x)| \neq 0$  for all  $x \in M$ . For these hypersurfaces, we have that the Gauss map can be written as  $\nu(x) = |\nabla f(x)|^{-1} \nabla f(x)$  for all  $x \in M$ . Clearly, the tangent space of  $M$  at a point  $x$  is the space of vectors  $v \in \mathbb{R}^n$  such that  $\langle v, \nabla f(x) \rangle = 0$ . Notice that the mean curvature of  $M$  at  $x$  is given by

$$-\sum_{i=1}^{n-1} \langle d\nu_x(v_i), v_i \rangle, \quad (1)$$

where  $\{v_1, \dots, v_{n-1}\}$  is an orthonormal bases of the vector space  $T_x M$ .

It is worth mentioning some elementary facts about real value functions on  $\mathbb{R}^n$  that will be used later on.

**Lemma 1.1.** *If  $f : V \rightarrow \mathbb{R}$  is a smooth function defined in an open set  $V$  of  $\mathbb{R}^n$ , then*

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$$(a) \quad \Delta f(x) = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = \sum_{i=1}^n \langle \text{Hess}(f)_x(w_i), w_i \rangle,$$

where  $\{w_1, \dots, w_n\}$  is any orthonormal basis of  $\mathbb{R}^n$  and  $\text{Hess}(f)$  is the  $n \times n$  Hessian matrix of  $f$ .

$$(b) \quad \langle \text{Hess}(f) \nabla f, \nabla f \rangle = \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle.$$

$$(c) \quad \frac{d \nabla f(\alpha(t))}{dt} = \text{Hess}(f)_{\alpha(t)} \alpha'(t) \text{ for any smooth curve } \alpha : (a, b) \rightarrow V.$$

*Proof.* (a) holds true because  $\Delta f(x)$  is the trace of the matrix  $\text{Hess}(f)_x$ , and the trace of a matrix is invariant under change of basis.

(b) Is a direct computation and (c) follows from the chain rule.  $\square$

## 2. Main result

In this section we will state and prove one of the main results of this paper.

**Theorem 2.1.** *Let  $M = \{x \in V : f(x) = 0\}$  where  $f : V \rightarrow \mathbb{R}$  is a smooth function defined in an open set  $V$  of  $\mathbb{R}^n$  with  $|\nabla f(x)| \neq 0$  for all  $x \in M$ .  $M$  is minimal if and only if  $|\nabla f|^2 \Delta f = \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle$  for every  $x \in M$ .*

*Proof.* We are going to compute the mean curvature  $H$  of  $M$  in terms of the function  $f$  and its partial derivatives. Let us start computing  $\langle d\nu_x(v), v \rangle$  for any  $v \in T_x M$ . Let us take a smooth curve  $\alpha : (-\epsilon, \epsilon) \rightarrow M$  such that  $\alpha(0) = x$  and  $\alpha'(0) = v$ . We have that

$$\begin{aligned} \langle d\nu_x(v), v \rangle &= \left\langle \frac{d\nu(\alpha(t))}{dt} \Big|_{t=0}, v \right\rangle \\ &= \left\langle \frac{d|\nabla f(\alpha(t))|^{-1} \nabla f(\alpha(t))}{dt} \Big|_{t=0}, v \right\rangle \\ &= \frac{d|\nabla f(\alpha(t))|^{-1}}{dt} \Big|_{t=0} \langle \nabla f(x), v \rangle + |\nabla f(x)|^{-1} \left\langle \frac{d\nabla f(\alpha(t))}{dt} \Big|_{t=0}, v \right\rangle \\ &= 0 + |\nabla f(x)|^{-1} \langle \text{Hess}(f)_x v, v \rangle = |\nabla f(x)|^{-1} \langle \text{Hess}(f)_x v, v \rangle. \end{aligned}$$

Now, if  $\{v_1, \dots, v_{n-1}\}$  is an orthonormal bases of  $T_x M$ , then by the equation (1) in

section 1, we get that

$$\begin{aligned} H &= - \sum_{i=1}^{n-1} \langle d\nu_x(v_i), v_i \rangle = -|\nabla f(x)|^{-1} \langle \text{Hess}(f)_x v_i, v_i \rangle \\ &= |\nabla f(x)|^{-1} (-\Delta f(x) + \langle \text{Hess}(f)_x |\nabla f(x)|^{-1} \nabla f(x), |\nabla f(x)|^{-1} \nabla f(x) \rangle) \\ &= |\nabla f(x)|^{-1} (-\Delta f(x) + |\nabla f(x)|^{-2} \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle). \end{aligned}$$

Therefore we get that  $M$  is minimal, if and only if, for every  $x \in M$ , we have that  $|\nabla f(x)|^2 \Delta f(x) = \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle$ .  $\square$

**Example 2.2 (Clifford minimal cones).** Let  $k$  and  $l$  be two positive integers such that  $k + l = n - 2$ , and let  $f : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$  be the function given by

$$f(x) = f(x_1, \dots, x_n) = k(x_1^2 + \dots + x_{l+1}^2) - l(x_{l+2}^2 + \dots + x_n^2).$$

Let us check that  $M_{lk} = f^{-1}(0)$  is a minimal hypersurface. A direct computation shows that

$$\begin{aligned} \nabla f(x) &= 2(kx_1, \dots, kx_{l+1}, -lx_{l+2}, \dots, -lx_n), \\ |\nabla f(x)|^2 &= 4k^2(x_1^2 + \dots + x_{l+1}^2) + 4l^2(x_{l+2}^2 + \dots + x_n^2), \\ \nabla |\nabla f(x)|^2 &= 8(k^2x_1, \dots, k^2x_{l+1}, l^2x_{l+2}, \dots, l^2x_n), \\ \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle &= 8k^3(x_1^2 + \dots + x_{l+1}^2) - 8l^3(x_{l+2}^2 + \dots + x_n^2), \\ \Delta f(x) &= 2k(l+1) - 2l(k+1) = 2k(k-l). \end{aligned}$$

Therefore, we have that if  $x \in M$ , i.e. if

$$k(x_1^2 + \dots + x_{l+1}^2) = l(x_{l+2}^2 + \dots + x_n^2),$$

then,

$$\begin{aligned} |\nabla f(x)|^2 \Delta f &= 2(k-l)(4k^2 - 4lk)(x_1^2 + \dots + x_{l+1}^2) \\ &= 8k(k^2 - l^2)(x_1^2 + \dots + x_{l+1}^2) \\ &= 8k^3x_1^2 + \dots + x_{l+1}^2 - 8l^3(x_{l+2}^2 + \dots + x_n^2) \\ &= \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle. \end{aligned}$$

We will say that  $M$  is a Clifford minimal cone if  $M = M_{kl}$  up to a rigid motion in  $\mathbb{R}^n$ .

Let us assume now that  $(N, g)$  is a riemannian  $n$  dimensional manifold,  $V$  is an open subset of  $N$  and  $f : V \rightarrow \mathbb{R}$  is a smooth function such that  $M = f^{-1}(0)$  is a hypersurface of  $N$ , i.e. 0 is a regular value of  $f$ . In this case we will denote by  $\text{Hess}(f)_x : T_x M \times T_x M \rightarrow \mathbb{R}$  the bilinear form defined by  $\text{Hess}(f)_x(v, w) = \langle D_v \nabla f, w \rangle$ , where  $D$  is the Levi-Civita connection on  $N$ .

The exact same proof of the previous theorem gives us the following result.

**Theorem 2.3.** *Let  $N$  be a riemannian manifold and let  $M = \{x \in V : f(x) = 0\}$ , where  $f : V \rightarrow \mathbb{R}$  is a smooth function defined in an open set  $V$  of  $N$  with  $|\nabla f(x)| \neq 0$  for all  $x \in M$ .  $M$  is minimal if and only if  $|\nabla f|^2 \Delta f = \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle$  for every  $x \in M$ .*

### 3. Minimal hypersurfaces of $\mathbb{R}^n$ given by quadratic form

In this section we characterize the minimal Clifford cones as the only hypersurfaces that are level sets of quadratic forms. More precisely we prove,

**Theorem 3.1.** *Let  $f : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$  be the function defined by  $f(x) = \langle Bx, x \rangle$ , where  $B$  is a  $n \times n$  symmetric matrix and take  $M = \{x \in \mathbb{R}^n : \langle Bx, x \rangle = 0\} \setminus \{0\} = f^{-1}(0)$ . The value 0 is a regular value of  $f$  and  $M$  is a minimal hypersurface if and only if  $M$  is a Clifford minimal cone.*

Before we prove this theorem we will need the following lemma.

**Lemma 3.2.** *Let  $B$  be an invertible symmetric matrix with both, positive and negative eigenvalues. If  $C$  is a matrix that commutes with  $B$  such that  $\langle Cx, x \rangle = 0$  whenever  $\langle Bx, x \rangle = 0$ , then  $C = \lambda B$  for some real number  $\lambda$ .*

*Proof.* Since  $B$  and  $C$  commutes, after an orthogonal change of coordinates, we can assume that

$$C = \begin{pmatrix} c_1 & & & & & \\ & \ddots & & & & \\ & & c_r & & & \\ & & & c_{r+1} & & \\ & & & & \ddots & \\ & & & & & c_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & & & & & \\ & \ddots & & & & \\ & & b_r & & & \\ & & & b_{r+1} & & \\ & & & & \ddots & \\ & & & & & b_n \end{pmatrix},$$

where  $0 < r < n$ ,  $b_1, \dots, b_r$  are positive real numbers and  $b_{r+1}, \dots, b_n$  are negative real numbers. Let us denote  $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$  the canonical bases for  $\mathbb{R}^n$ , and for  $1 \leq i \leq r$  and  $r < j \leq n$  we will denote  $x_{ij} = \sqrt{-b_j}e_i + \sqrt{b_i}e_j$ . Note that  $\langle Bx_{ij}, x_{ij} \rangle = 0$ , therefore  $\langle Cx_{ij}, x_{ij} \rangle = 0$ , i.e.,

$$-b_j c_i + b_i c_j = 0;$$

from the equation above we get that  $c_i = \frac{c_n}{b_n} b_i$  for  $1 \leq i \leq r$ , and for  $r < j \leq n$  we have, using the expression for  $c_1$ ,

$$c_j = \frac{c_1}{b_1} b_j = \frac{c_n}{b_n} b_1 \frac{1}{b_1} b_j = \frac{c_n}{b_n} b_j.$$

Therefore  $C = \frac{c_n}{b_n} B$ , and this completes the proof.  $\square$

Now we are ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* A Direct computation shows that  $\nabla f(x) = 2Bx$ . Since we are assuming that 0 is a regular value, then  $\nabla f(x) \neq \mathbf{0}$  for all  $x \in M$ ; in particular,  $Bx_0 \neq \mathbf{0}$  for every  $x_0 \neq \mathbf{0}$ , because if  $Bx_0 = \mathbf{0}$  for some  $x_0 \neq \mathbf{0}$  we would have that  $x_0 \in M$  and  $\nabla f(x_0) = \mathbf{0}$ . Therefore  $B$  is an invertible matrix. We also have that, since  $M \neq \emptyset$ , then  $B$  must have positive eigenvalues and negative eigenvalues. We have them

$$|\nabla f(x)|^2 = \langle 2Bx, 2Bx \rangle = 4\langle Bx, Bx \rangle = 4\langle B^2x, x \rangle.$$

Since  $B^2$  is symmetric, then  $\nabla|\nabla f|^2(x) = 8B^2x$ . A direct computation shows that  $\Delta f = 2\text{trace}(B)$ . Using Theorem 2.1, we have that  $M$  is minimal if and only if  $|\nabla f|^2 \Delta f = \frac{1}{2} \langle \nabla|\nabla f|^2, \nabla f \rangle$  for every  $x \in M$ , i.e. if, for every  $x \neq \mathbf{0}$  such that  $\langle Bx, x \rangle = 0$  we have that

$$4\langle B^2x, x \rangle (2\text{trace}(B)) = \frac{1}{2} \langle 8B^2x, 2Bx \rangle = 8\langle B^3x, x \rangle.$$

In others words, if we define  $C = \text{trace}(B)B^2 - B^3$ , we have that  $M$  is minimal if and only if  $\langle Cx, x \rangle = 0$  for every  $x$  such that  $\langle Bx, x \rangle = 0$ . Using Lemma 3.2 we concluded that there exists a real number  $a$  such that  $\text{trace}(B)B^2 - B^3 = aB$ . Since  $B$  is an invertible matrix, we get that  $B$  satisfies the polynomial equation

$$B^2 - \text{trace}(B)B + aI = 0. \quad (2)$$

Therefore  $B$  can only have two eigenvalues. Since  $B$  has negative and positive eigenvalues, we can assume that the eigenvalues of  $B$  are  $\lambda_1 > 0$  with multiplicity  $r \geq 1$  and  $\lambda_2 < 0$  with multiplicity  $n - r \geq 1$ . Note that  $\text{trace}(B) = r\lambda_1 + (n - r)\lambda_2$ . The equation (2) is equivalent to the following system of equations for  $\lambda_1, \lambda_2$  and  $a$ :

$$\begin{aligned} \lambda_1^2 - (r\lambda_1 + (n - r)\lambda_2)\lambda_1 + a &= (1 - r)\lambda_1^2 - (n - r)\lambda_1\lambda_2 + a = 0, \\ \lambda_2^2 - (r\lambda_1 + (n - r)\lambda_2)\lambda_2 + a &= -(n - r - 1)\lambda_2^2 - r\lambda_1\lambda_2 + a = 0; \end{aligned}$$

combining these two equations we get

$$(1 - r)\lambda_1^2 - (n - 2r)\lambda_1\lambda_2 + (n - r - 1)\lambda_2^2. \quad (3)$$

From this equation we get that  $r = 1$  or  $r = n - 1$  implies that  $\lambda_1 = \lambda_2$  which is impossible because  $\lambda_1\lambda_2 < 0$ . Therefore  $1 < r < n - 1$ . From equation (3) we get that  $t = \frac{\lambda_2}{\lambda_1}$  satisfies the equation

$$(1 - r) - (n - 2r)t + (n - r - 1)t^2.$$

Therefore,  $t = 1$  or  $t = \frac{r - 1}{n - r - 1}$ ; since we have that  $t$  must be negative, then  $t$  cannot be 1. Therefore, up to a constant we may take  $\lambda_1 = n - r - 1$  and  $\lambda_2 = r - 1$ . I.e., up to a rigid motion  $f(x) = \langle Bx, x \rangle$  must be a multiple of the function given in the example 2.2. This implies that  $M$  must be a Clifford minimal cone.  $\square$

**Remark on the construction of minimal hypersurfaces using homogeneous polynomials of degree  $k$ :** Let  $f : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$  be a homogeneous polynomial of degree  $k$  such that  $f^{-1}(0) = M$  is not empty and such that for every  $x \in M$ ,  $\nabla f(x) \neq 0$ . By theorem 2.1 we have that  $M$  is minimal if and only if

$$g(x) = |\nabla f|^2 \Delta f - \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle = 0 \quad (4)$$

for every  $x$  such that  $f(x) = 0$ . Notice that the left hand side of the equation (4) is a homogeneous polynomial of degree  $3k - 4$ ; also notice that if  $g(x) = h(x)f(x)$  for some homogeneous polynomial  $h$  of degree  $2k - 4$ , then  $M$  will be minimal.

It is easy to prove the veracity of the following:

**Conjecture.** Let  $f : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$  be a homogeneous polynomial of degree  $k$  such that  $f^{-1}(0) = M$  is not empty and such that for every  $x \in M$ ,  $\nabla f(x) \neq 0$ . If  $g(x)$  is a polynomial of degree  $m$  with  $m \geq k$  such that  $g(x) = 0$  for every  $x \in M$ , then there exists a homogeneous polynomial  $h$  of degree  $m - k$  such that  $g(x) = h(x)f(x)$ .

That conjecture implies the following result:

“Let  $M = f^{-1}(0) \neq \emptyset$ , where  $f : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$  is a homogeneous polynomial.  $M$  is minimal if and only if

$$|\nabla f|^2 \Delta f - \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle = hf$$

for some homogeneous polynomial  $h$ ”.

So far the only known examples of these minimal hypersurfaces are the isoparametric minimal hypersurfaces; the degree of  $f$  in these examples are  $k = 1, 2, 3, 4, 6$ . Note that Theorem 2.1 proves the conjecture when  $k = 2$ .

## References

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